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ON THE NON-ESCAPING SET OF $e^{2\pi i\theta}\sin(z)$

BY

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ABSTRACT

Let $0 < \theta < 1$ be an irrational number of bounded type. We proved that almost every point in the Julia set of $e^{2\pi i\theta}\sin(z)$ is an escaping point.

1. Introduction

Let $0 < \theta < 1$ be an irrational number of bounded type. It was proved in [Z] that the boundary of the Siegel disk for the map $f_\theta(z) = e^{2\pi i\theta}\sin(z)$ is a quasi-circle which passes through exactly two critical points $\pi/2$ and $-\pi/2$. Let J_θ denote the Julia set of f_θ. We call the set

$$I_\theta = \{z \in \mathbb{C} : f_\theta^n(z) \to \infty \text{ as } n \to \infty\}$$

the escaping set and its complement

$$K_\theta = J_\theta - I_\theta$$

the non-escaping set. By a theorem of McMullen [McM], the Lebesgue measure of the escaping set I_θ is always positive. The main result of this paper is

Main Theorem: *Let $0 < \theta < 1$ be an irrational number of bounded type. Then the Lebesgue measure of K_θ is zero. That is to say, the forward iteration of almost every point in J_θ escapes to the infinity.*

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There is some similarity between the Main Theorem and Petersen's zero measure result on the Julia set of a quadratic polynomial with a bounded type Siegel disk [P1]. In fact, the idea used there also plays a role here. For instance, Lemma 3.2 in this paper, which is one of the key lemmas in the whole proof, is a variant of Lemma 1.11 of [P1]. The difference between these two situations lies in the following two facts:

1. The Blaschke product, which is one of the main tools used in [P1], does not exist in our case.
2. There are no external rays and equipotential curves for f_θ, and therefore, the puzzle method used in [P1] cannot be applied here.

To solve the first problem, we will construct a symmetric model map $F_\theta(z)$, which will play the same role as the Blaschke product does in the case of quadratic Siegel polynomials. To solve the second one, we will introduce a new geometric object, called **minimal neighborhood**, by which we can do all the necessary analysis without using the puzzles.

The organization of the paper is as follows. In §2, we will outline the proof in [Z] first and then construct the model map F_θ. In §3, we will prove that the inverse branch of F_θ contracts the hyperbolic metric in some appropriate hyperbolic Riemann surface. In §4, we will introduce the concept of the **minimal neighborhood**, and prove some basic properties of this object. In §5, we prove the Main Theorem by using the pull back argument.

2. The Model Map $F_\theta(z)$

The goal of this section is to construct the symmetric model map $F_\theta(z)$. The construction is based on the line of the proof in [Z]. So let us sketch the proof in [Z] first, and then turn to the construction of F_θ at an appropriate point. Let $0 < \theta < 1$ be a bounded type irrational number and be fixed throughout the following. Let $\Delta = \{z : |z| < 1\}$ denote the open unit disk. Set

$$g(z) = \sin(z)/2.$$

Clearly $g(z)$ has an attracting fixed point at the origin with multiplier $1/2$. Let Ω be the maximal domain centered at the origin on which g is holomorphically conjugate to the linear map $z \to z/2$.

LEMMA 2.1 (Lemmas 2 and 3 in [Z]): *The domain Ω is bounded and symmetric about the origin, and, moreover, $\partial\Omega$ is a quasi-circle which passes through exactly two critical points $\pi/2$ and $-\pi/2$.*

Let $\gamma' = g(\partial\Omega)$. It follows that γ' is a real-analytic curve. Let $c_1 = \pi/2$ and $\xi = g(c_1)$. Then for each $\eta \in \partial\Omega$, by the Riemann mapping theorem, there is a unique holomorphic isomorphism

$$\mu_\eta : \widehat{\mathbb{C}} - \overline{g(\Omega)} \to \widehat{\mathbb{C}} - \overline{\Omega}$$

such that $\mu_\eta(\infty) = \infty$ and $\mu_\eta(\xi) = \eta$. Clearly μ_η can be homeomorphically extended to $\partial g(\Omega) = g(\partial\Omega)$. The map μ_η is odd ([Z, Lemma 4]). Moreover, we have

LEMMA 2.2 ([Z, Lemma 5]): *There exists a unique $\eta \in \partial\Omega$ such that*

$$\mu_\eta \circ g|\partial\Omega : \partial\Omega \to \partial\Omega$$

is a topological circle homeomorphism of rotation number θ.

Let $g_\eta = \mu_\eta \circ g$. Let

$$\psi : \widehat{\mathbb{C}} - \Delta \to \widehat{\mathbb{C}} - \Omega$$

be the holomorphic isomorphism such that $\psi(\infty) = \infty$ and $\psi(1) = c_1$. The map ψ is odd([Z, Lemma 7]).

LEMMA 2.3 ([Z, Lemma 8]): *The circle homeomorphism*

$$f = \psi^{-1} \circ g_\eta \circ \psi : \partial\Delta \to \partial\Delta$$

can be extended to an analytic critical circle mapping such that f has exactly two double critical points at 1 and -1.

By Herman–Swiatek's linearization theorem on critical circle mappings [P2], it follows that there is a unique quasi-symmetric circle homeomorphism h such that $h(1) = 1$ and $f = h \circ R_\theta \circ h^{-1}$. The map h is odd([Z, Lemma 10]).

LEMMA 2.4 ([Z, Lemma 11]): *The maps μ_η, ψ, and h can be respectively extended to quasiconformal homeomorphisms $\tilde{\mu}_\eta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $\tilde{\psi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and $H : \Delta \to \Delta$, and moreover, all these maps are odd.*

Now for each $k \in \mathbb{Z}$, let $\Omega_k = \{z : z + k\pi, z \in \Omega\}$. The sets $\Omega_k, k \in \mathbb{Z}$ are pairwise disjoint ([Z, Lemma 12]). Define

$$\tilde{f}_\theta(z) = \begin{cases} (\tilde{\mu}_\eta \circ g)(z) & \text{for } z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \Omega_k, \\ \tilde{\psi} \circ H \circ R_\theta \circ H^{-1} \circ \tilde{\psi}^{-1}(z - k\pi) & \text{for } z \in \Omega_k,\ k \text{ is even}, \\ -\tilde{\psi} \circ H \circ R_\theta \circ H^{-1} \circ \tilde{\psi}^{-1}(z - k\pi) & \text{for } z \in \Omega_k,\ k \text{ is odd}. \end{cases} \tag{1}$$

Lemma 2.5: *The map $\tilde{f}_\theta(z)$ is quasiconformally conjugate to $f_\theta(z)$.*

For a detailed proof of Lemma 2.5, see [Z]. Set

$$X = \bigcup_{k\in\mathbb{Z}, k\neq 0} \psi^{-1}(\Omega_k). \tag{2}$$

For $z \in \mathbb{C}$, let us use z^* to denote the symmetric image of z about the unit circle. Let $X = \{z^* : z \in X\}$. Define the model map

$$F_\theta(z) = \begin{cases} \psi^{-1} \circ \mu_\eta \circ g \circ \psi(z) & \text{for } |z| \geq 1 \text{ and } z \notin X, \\ (\psi^{-1} \circ \mu_\eta \circ g \circ \psi(z^*))^* & \text{for } z < 1 \text{ and } z \notin X^*. \end{cases} \tag{3}$$

Lemma 2.6: *The map $F_\theta(z) : \mathbb{C} - (X \cup X^*) \to \mathbb{C}$ is holomorphic. Moreover, F_θ can be extended to a symmetric and holomorphic map in a neighborhood of the unit circle, such that 1 and -1 are the two double critical points of F_θ.*

Proof. The first assertion is implied by the construction. The second assertion follows from Lemma 2.3. ∎

Let D be the Siegel disk of f_θ. By Lemma 2.5, and the definition of F_θ, we have

Lemma 2.7: *There is a quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ such that*

1. $\phi(\infty) = \infty$, $\phi(\pi/2) = 1$, *and* $\phi(-\pi/2) = -1$,
2. $\phi(f_\theta^{-1}(D)) = X \cup \Delta$,
3. $\phi^{-1} \circ F_\theta \circ \phi(z) = f_\theta(z)$ *for all* $z \in \mathbb{C} - f_\theta^{-1}(D)$.

3. Contraction of F_θ^{-1}

Lemma 3.1: *Let X be the set defined by (2). Then $\mathbb{C} - \overline{X \cup \Delta}$ has exactly two components so that one is above the other one.*

The proof is easy and we leave it to the reader. Let us use Ω_+ to denote the upper one and Ω_- to denote the lower one. Let $\Omega_* = \mathbb{C} - \overline{\Delta}$. It follows that $F_\theta : \Omega_+ \to \Omega_*$ and $F_\theta : \Omega_- \to \Omega_*$ are both holomorphic covering maps. Let us use $d\rho_* = \lambda_{\Omega_*}|dz|$, $d\rho_+ = \lambda_{\Omega_+}|dz|$ and $d\rho_- = \lambda_{\Omega_-}|dz|$ to denote the hyperbolic metrics on Ω_*, Ω_+, and Ω_- respectively.

Take $c \in \{1, -1\}$. To fix the idea, assume that $c = 1$. Take $r > 0$ small. Let $B_r(c)$ denote the Euclidean disk centered at c and with radius r. Since F_θ can be holomorphically extended to a neighborhood of the unit circle (Lemma 2.6), it follows that when $r > 0$ is small enough, there are exactly two domains which are contained in $B_r(c) \cap \{z \bigm| |z| > 1\}$, and which are mapped into the outside of the unit disk by F_θ. Moreover, one of them is contained in Ω_+, and the other one is contained in Ω_-. In addition, there exists another domain, say U_c, which is mapped into the inside of the unit disk by F_θ. Since F_θ is locally a $3:1$ holomorphic branched covering map at c(Lemma 2.6), it follows that the two angles formed by the unit circle and the two boundary segments of U_c are both equal to $\pi/3$(see Figure 1). Take $0 < \epsilon < 1/12$. Let R and L be the two rays starting from c such that the angles between $\partial\Delta$ and R, $\partial\Delta$ and L, are both equal to $\epsilon\pi$. Let S_ϵ^c be the cone spanned by R and L which is attached to c from the outside of the unit disk. Set

$$\Omega_{\epsilon,r}^c = S_\epsilon^c \cap (\Omega_+ \cup \Omega_-) \cap B_r(c). \tag{4}$$

The following lemma says that on $\Omega_{\epsilon,r}^c$, F_θ strictly increases the hyperbolic metric in Ω_*. The lemma is a variant of Lemma 1.11 in [P1].

Lemma 3.2: *There is a $\delta > 0$ such that for any $x \in \Omega_{\epsilon,r}^c$, we have*

$$\lambda_{\Omega_*}(F_\theta(x))|F_\theta'(x)| \geq (1+\delta)\lambda_{\Omega_*}(x).$$

Proof. Take any point $x \in \Omega_{\epsilon,r}^c$. Without loss of generality, we may assume that $x \in \Omega_+$. Since $F_\theta : \Omega_+ \to \Omega_*$ is a holomorphic covering map, we have

$$\lambda_{\Omega_*}(F_\theta(x))|F_\theta'(x)| = \lambda_{\Omega_+}(x).$$

So we need only to prove that there exists a uniform $\delta > 0$ such that

$$\lambda_{\Omega_+}(x)/\lambda_{\Omega_*}(x) \geq 1 + \delta.$$

Since $\Omega_+ \subset \Omega_*$, it is sufficient to prove the above inequality for $x \in B_r(c) \cap \Omega_{\epsilon,r}^c$. In fact, if r is small, when viewed from the point x, Ω_+ is approximately an angle domain with the vertex c, and with angle $\pi/3$. By taking an appropriate

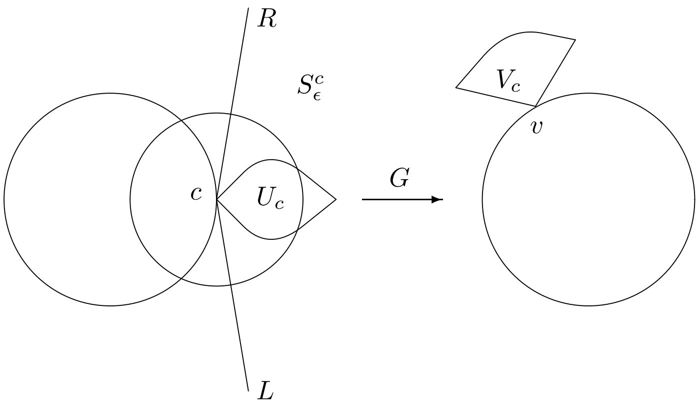

Figure 1. The contraction region of F_θ^{-1}.

coordinate system, we may write $x = c + \eta e^{i\lambda\pi}$ where $\epsilon < \lambda < 1/3$, and $0 < \eta < r$. Thus we get

$$\lambda_{\Omega_+}(x) \approx 3\eta^2 \frac{1}{\eta^3 \sin 3\lambda\pi} = \frac{3}{\eta \sin 3\lambda\pi}.$$

On the other hand, when viewed from x, Ω_* is approximately the half plane, therefore,

$$\lambda_{\Omega_*}(x) \approx \frac{1}{\eta \sin \lambda\pi}.$$

This gives us

$$\lambda_{\Omega_+}(x)/\lambda_{\Omega_*}(x) \approx \frac{3 \sin \lambda\pi}{\sin 3\lambda\pi} > \frac{3 \sin \epsilon\pi}{\sin 3\epsilon\pi} > 1. \quad \blacksquare$$

4. Minimal Neighborhoods

Let D be the Siegel disk of f_θ centered at the origin. It follows that ∂D is a quasi-circle passing through $\pi/2$ and $-\pi/2$, [Z].

4.1. Hyperbolic Neighborhoods. Let $I \subset \partial D$ be a curve segment. Set

$$\Omega_I = \mathbb{C} - (\partial D - I).$$

It follows that Ω_I is a hyperbolic Riemann surface. For any $x, y \in \Omega_I$, let $d_{\Omega_I}(x, y)$ denote the distance between x and y with respect to the hyperbolic metric on Ω_I. For a given $d > 0$, set

$$H_d(I) = \{z \in \Omega_I : d_{\Omega_I}(z, I) < d\}.$$

One can regard $H_d(I)$ as a variant of the hyperbolic neighborhood in the slit complex plane (see §5 of Chapter VI, [MS]).

Remark 4.1: The hyperbolic neighborhood $H_d(I)$ is a simply connected domain when both I and $d > 0$ are small. More precisely, there is an $\epsilon_0 > 0$ and a $d_0 > 0$ such that for any curve segment $I \subset \partial D$ with Euclidean diameter less than ϵ_0, and any quantity $0 < d < d_0$, the hyperbolic neighborhood $H_d(I)$ is a simply connected domain. The proof is direct and we shall leave it to the reader.

Throughout the following, we always assume that d and I involved are small enough so that $H_d(I)$ is a simply connected domain.

LEMMA 4.1: *Let $I \subset \partial D$ be an arc segment which contains neither $e^{2\pi i\theta}$ nor $-e^{2\pi i\theta}$. Let $J \subset \partial D$ be the curve segment such that $f_\theta(J) = I$. Let V be the connected component of $f_\theta^{-1}(H_d(I))$ which contains J. Then $V \subset H_d(J)$.*

Proof. Since I does not contain the critical values of f_θ, it follows that J does not contain the critical points of f_θ. This then implies that there exist exactly two components of $f_\theta^{-1}(\Omega_I)$ so that one is above the other. Let us use Q_J^+ to denote the upper one and Q_J^- to denote the lower one (see Figure 2). It follows that V is contained in exactly one of them. Without loss of generality, we may assume that $V \subset Q_J^-$. The Lemma then follows from the fact that $f_\theta : Q_J^- \to \Omega_I$ is a holomorphic covering map and $Q_J^- \subset \Omega_J$. ∎

4.2. THE DYNAMICAL LENGTH. Let $\sigma : \partial D \to \partial\Delta$ be the homeomorphism such that $f_\theta|\partial D = \sigma^{-1} \circ R_\theta \circ \sigma$ and $\sigma(\pi/2) = 1$. Then for any curve segment $I \subset \partial D$, define $l(I)$ to be the Euclidean length of the arc segment $\sigma(I)$. It follows that l is f_θ-invariant. Namely,

$$l(I) = l(f_\theta(I))$$

holds for any $I \subset \partial D$. We call $l(I)$ the dynamical length of I.

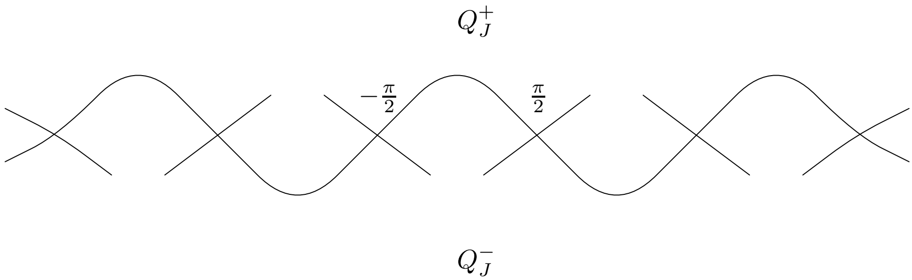

Figure 2. The two components Q_J^+ and Q_J^-.

4.3. Minimal Neighborhoods. Let us fix $d > 0$ through the following discussions. Let $z_0 \in \mathbb{C} - \bigcup_{k=0}^{\infty} f_\theta^{-k}(\overline{D})$. Let $z_k = f_\theta^k(z_0)$. Assume that $z_k \to \partial D$ as $k \to \infty$. For each $k \geq 0$, define

$$\Phi_k = \{I \subset \partial D : z_k \in \overline{H_d(I)}\}.$$

Define

$$l_k = \inf\{l(I) : I \in \Phi_k\}.$$

By using compact argument, one can easily prove the following

Lemma 4.2: *For every $k \geq 0$, there exists a curve segment $I \in \Phi_k$ such that $l(I) = l_k$.*

The minimal curve segment I in Lemma 4.2 may not be unique. To fix the idea, let us use I_k to denote one of them. Since $z_k \to \partial D$ and ∂D is a quasi-circle, it follows that

Lemma 4.3: *$l_k \to 0$ as $k \to \infty$.*

For each $n \geq 0$, let $\lambda_n = \min\{l_k : 0 \leq k \leq n\}$. Define

$$m(n) = \min\{0 \leq k \leq n : l_k = \lambda_n\}.$$

From the definition of $m(n)$ and Lemma 4.3, we have

Lemma 4.4: *$m(n) \leq m(n+1)$ and $m(n) \to \infty$ as $n \to \infty$.*

For each $m(n)$, we call the hyperbolic neighborhood $H_d(I_{m(n)})$ a **minimal neighborhood**.

4.4. The Dynamics of the Model Map F_θ. Let $I \subset \partial\Delta$. We use $|I|$ to denote the Euclidean length of I. Let

$$W_I = \mathbb{C} - (\partial\Delta - I) \quad \text{and} \quad \widehat{W_I} = \widehat{\mathbb{C}} - (\partial\Delta - I).$$

We use d_{W_I} and $d_{\widehat{W_I}}$ to denote the hyperbolic metrics on W_I and $\widehat{W}$, respectively. For $d > 0$, set

$$U_d(I) = \{z \in W_I : d_{W_I}(z, I) < d\}$$

and

$$\widehat{U}_d(I) = \{z \in \widehat{W}_I : d_{\widehat{W_I}}(z, I) < d\}.$$

Since $\widehat{U}_d(I)$ is isomorphic to the hyperbolic neighborhood in the slit complex plane, it follows that $\widehat{U}_d(I)$ is bounded by two arcs of Euclidean circles which are symmetric about the unit circle, and moreover, the exterior angles between the arcs and the unit circle, which are between 0 and π, are uniquely determined by d. The reader may refer to [MS] (§5, Chapter VI) for further knowledge about this object. The following lemma follows easily from the fact that $\widehat{W} = W \cup \{\infty\}$ and we leave the details to the reader.

Lemma 4.5: *For any $d > 0$, there exist $0 < \lambda(d) < d$ and $\epsilon > 0$ such that for all $I \subset \partial\Delta$ with $|I| < \epsilon$, the following relation holds,*

$$\widehat{U}_{\lambda(d)}(I) \subset U_d(I) \subset \widehat{U}_d(I).$$

Let ϕ be the quasiconformal homeomorphism defined in Lemma 2.7.

Lemma 4.6: *For any $d > 0$, there exist $0 < d' < d''$ which depends only on d such that for any $I \subset \partial D$, the following relation holds,*

$$U_{d'}(\phi(I)) \subset \phi(H_d(I)) \subset U_{d''}(\phi(I)).$$

Proof. This follows from the fact that the distortion of the hyperbolic metric by a quasiconformal homeomorphism ϕ is bounded by a number which only depends on the dilation of ϕ [LV] (see §3.3 of Chapter II). ∎

Since $F_\theta|\partial\Delta$ is quasi-symmetrically conjugate to the rigid rotation R_θ, for each arc $I \subset \partial\Delta$, as in §4.2, we may define the dynamical length $L(I)$ as follows. Let $h : \partial\Delta \to \partial\Delta$ be the quasi-symmetric homeomorphism such that $h^{-1} \circ F_\theta \circ h = R_\theta$ and $h(1) = 1$. For each $I \subset \partial\Delta$, define $L(I)$ to be the

Euclidean length of the arc $h^{-1}(I)$. We call $L(I)$ the dynamical length of I. Clearly, $L(I)$ is F_θ-invariant in the sense

$$L(I) = L(F_\theta(I)).$$

Note that we have defined the dynamical length in two models, one is on the boundary of the Siegel disk, and the other is on the unit circle. From the definitions, we get

LEMMA 4.7: *Let $I \subset \partial D$ be a curve segment. Then $l(I) = L(\phi(I))$ where ϕ is the map defined in Lemma 2.7.*

LEMMA 4.8: *For any $0 < \delta < 1$, there exists an $\epsilon > 0$ such that for any $I \subset \partial\Delta$ and $J \subset \partial\Delta$, if $I \cap J \neq \emptyset$ and $|J| < \epsilon|I|$, then $L(J) < \delta L(I)$.*

Proof. Since h is quasi-symmetric, it follows that h^{-1} is quasi-symmetric also. Let $1 < M < \infty$ be the quasi-symmetric constant of h^{-1}. From the definition of the dynamical length L, it follows that for any two adjacent intervals S and T in $\partial\Delta$ which have equal Euclidean length, we have

$$L(S)/M \leq L(T) \leq ML(S).$$

Now let us prove the lemma. By taking I to be $I - J$ if $I - J$ is connected and the longer component of $I - J$ otherwise, we may assume that J is not contained in I, and that I and J have one common end point. For $0 < \epsilon < 1$ small, let N be the integer part of the quantity $\log_2(1/\epsilon)$. Since I and J have a common end point, we can take 2^N adjacent intervals, say $J_1, J_2, \ldots, J_{2^N}$, in I such that: (1) all of them have the same Euclidean length as J, and (2) the intervals J_1 and J have one common end point.

Now let us deduce the lemma as follows. First we have

$$L(J_1) \geq L(J)/M.$$

Since J_1 and J_2 are adjacent to each other and have the same Euclidean length, we get

$$L(J_1 \cup J_2) \geq (1 + 1/M)L(J_1) \geq (1 + 1/M)L(J)/M.$$

Since $J_1 \cup J_2$ and $J_3 \cup J_4$ are adjacent to each other and have the same Euclidean length, we get

$$L(J_1 \cup J_2 \cup J_3 \cup J_4) \geq (1 + 1/M)L(J_1 \cup J_2) \geq (1 + 1/M)^2 L(J)/M.$$

Repeating this procedure, we finally get

$$L(I) \geq L\Big(\bigcup_{1\leq k\leq 2^N} J_k\Big) \geq (1+1/M)^N L(J)/M.$$

This implies Lemma 4.8. ∎

Lemma 4.9: *Let $d>0, d'>0$ and $0<\eta<1$. Then there exist $\delta>0$ and $\epsilon>0$, which depend on d, d' and η, such that for any $I\subset\partial\Delta$ with $|I|<\epsilon$, and any $x\in\overline{U_d(I)}$, if there exists a point $y\in I$ such that the angle between the straight segment $[x,y]$ and the unit circle is less than δ, then there exists $J\subset\partial\Delta$ such that $x\in U_{d'}(J)$ and $L(J)<\eta L(I)$.*

Remark 4.2: There are two places where we require that $|I|$ be small. The first one is that we will regard I as a straight segment in the following estimations of the geometry. This will not affect the validity of the proof since the errors caused by such approximation is negligible. The second one is that we will apply Lemma 4.5 (the left hand of the inclusion) to d' and the interval J.

Proof. Let $\epsilon>0$ be small and $I\subset\partial\Delta$ be an arc such that $|I|<\epsilon$. Since $U_d(I)\subset\widehat{U}_d(I)$, it follows that there is an $0<\alpha<\pi$ which depends only on d such that (1) there exist two rays L and R such that the exterior angles between the horizontal line and the two rays L and R are both equal to α, and (2) the part of the neighborhood $U_d(I)$, which lies in the outside of the unit disk, is between L and R.

Let a and b be the two end points of I. let c be a point in $U_d(I)$ such that $|ac|=|bc|$. Let d be the intersection point of L and the straight line which passes through b and c. Similarly, let e be the intersection point of R and the straight line which passes through a and c. Let $\delta=\angle cab=\angle cba$ (see Figure 3). It is sufficient to show that the property in the lemma is true when $\delta>0$ and $\epsilon>0$ are both small enough. First, let us assume that $0<\delta<\pi/8$ and $0<\epsilon<\pi/24$.

Let $x\in\overline{U}_d(I)$. By the symmetry, we may assume that x lies outside the unit disk. Then there are only two cases. In the first case, x lies in the triangle Δ_{abc}. In the second case, x lies in either the triangle Δ_{acd} or the triangle Δ_{bce} (This is because for other x, the angle between $[x,y]$ and I will be greater than δ.)

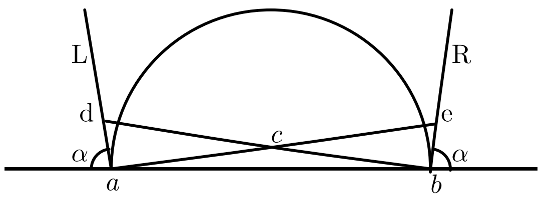

Figure 3. The two rays L and R and the segments $[a, e]$ and $[b, d]$.

In the first case, by Lemma 4.8, it follows easily that for the given $\eta > 0$, when $\delta > 0$ is small enough, there exists an interval $J \subset \partial\Delta$ which satisfies the properties in the lemma.

Now let us assume that we are in the second case. We need only to consider the case that x belongs to the triangle Δ_{acd}. (The same argument will apply to the triangle Δ_{bce} by symmetry.) Let $w \in I$ such that

$$|wx| = \inf_{z \in I}\{|zx|\}.$$

Note that if the projection of x on the horizontal line lies in the outside of I, then $w = a$ (see Figure 4). It follows that

$$\pi/2 \le \angle xwb \le \pi - \alpha \text{ if } \alpha < \pi/2 \quad \text{and} \quad \angle xwb = \pi/2 \text{ if } \alpha \ge \pi/2, \tag{5}$$

and hence that

$$\alpha - \delta \le \angle bxw < \pi/2 \text{ if } \alpha < \pi/2 \quad \text{and} \quad \pi/2 - \delta \le \angle bxw < \pi/2 \text{ if } \alpha \ge \pi/2. \tag{6}$$

By the assumption that $0 < \epsilon < \pi/24$ and $0 < \delta = \angle cba < \pi/8$, we get $|I| < \pi/24$, and

$$|wx| \le |ad| \le C \tag{7}$$

where $0 < C < \infty$ is some constant dependening only on α, and hence on d. For $d' > 0$, let $0 < \lambda(d') < d'$ be the quantity in Lemma 4.5. Let J be the shortest interval with respect to the Euclidean metric such that

i. $x \in \overline{\widehat{U}_{\lambda(d')/2}(J)}$, and
ii. w is the mid-point of J.

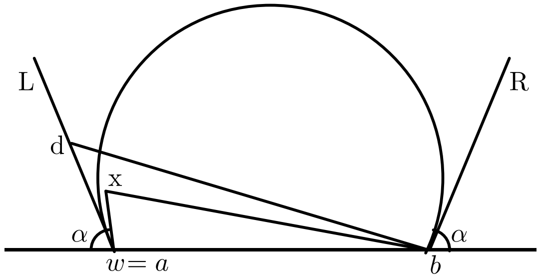

Figure 4. $w = a$ when the projection of x on the horizontal line lies in the outside of I.

It follows from (i), (ii), (5) and (7) that

$$|J| \leq C'|wx|$$

where $0 < C' < \infty$ is some constant which depends only on d' and d. On the other hand, from (6), we get that

$$\sin \angle bxw \geq 1/C''$$

holds, provided $\delta > 0$ is small enough, where $0 < C'' < \infty$ is some constant which depends only on α, and hence on d. Now applying the Sine Law to the triangle Δxwb, we get that

$$|wx| = |bw|\frac{\sin \angle xbw}{\sin \angle bxw} \leq \frac{\delta |I|}{\sin \angle bxw} \leq C''\delta|I|.$$

It follows that

$$|J| \leq C'|wx| \leq C'C''\delta|I|.$$

The above inequality implies two things: First, that when $\epsilon > 0$ is small enough, we get $|I| < \epsilon$ is small, and therefore $|J|$ is small, and so we can apply Lemma 4.5 (the left hand of the inclusion) and get

$$x \in \overline{\widehat{U}_{\lambda(d')/2}(J)} - \partial J \subset \widehat{U}_{\lambda(d')}(J) \subset U_{d'}(J).$$

Second, when $\delta > 0$ is small enough, we can make sure that $L(J) < \eta L(I)$ by Lemma 4.8. This completes the proof of the lemma. ∎

Let $z_0 \in \mathbb{C} - \bigcup_{k=0}^{\infty} f_\theta^{-k}(\overline{D})$ such that $z_k \to \partial D$. Set

$$z_k = f^k(z_0), \text{ and } \omega_k = \phi(z_k)$$

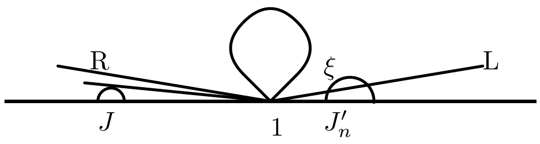

Figure 5. Case 1.

where $\phi : \mathbb{C} \to \mathbb{C}$ is the quasi-conformal homeomorphism defined in Lemma 2.7. From Lemma 2.7, it follows that

Lemma 4.10: *$F_\theta^k(\omega_0) = \omega_k$ and $\omega_k \to \partial\Delta$.*

Lemma 4.11: *Let $r > 0$ be small. Then there exists an $0 < \epsilon < 1/12$ such that $\omega_{m(n)-1} \in \Omega_{\epsilon,r}^c$ for all n large enough, where $c \in \{-1, 1\}$ and $\Omega_{\epsilon,r}^c$ is the domain defined in* §3.

Proof. We divide the proof into two cases.

Case 1. The minimal arc $I_{m(n)} \subset \partial D$ contains neither $e^{2\pi i\theta}$ nor $-e^{2\pi i\theta}$. Let $J_n \subset \partial D$ be the arc such that $I_{m(n)} = f_\theta(J_n)$. By Lemma 4.1 and the minimal property of $m(n)$, it follows that $z_{m(n)}$ has another pre-image, which is distinct from $z_{m(n)-1}$ and which is contained in $H_d(J_n)$, and is therefore close to ∂D also. Let us denote this point by z. Now it is important to note the following fact: for each $t > 0$ sufficiently small, the image by f_θ of the t-neighborhood of D, say D_t, covers any point of its image at most twice, and moreover, each critical value has a neighborhood which is evenly double covered by some neighborhood in D_t of the appropriate critical point. This implies that $z_{m(n)-1}$ and z are contained in a small neighborhood of some critical point on the boundary of the Siegel disk. Moreover, since z_k approaches to ∂D as $k \to \infty$, the neighborhood can be arbitrarily small provided n is large enough. Without loss of generality, we may assume that this critical point is $\pi/2$.

Now let us transfer these data to the dynamical plane of F_θ by the quasi-conformal map ϕ defined in Lemma 2.7. Recall that $\omega_{m(n)-1} = \phi(z_{(m(n)-1)})$ and $\phi(\pi/2) = 1$. Let $\xi = \phi(z)$ and $J_n' = \phi(J_n)$. It follows that both $\omega_{m(n)-1}$

and ξ belong to $B_r(1)$ for all n large enough. Since $F_\theta(\omega_{m(n)-1}) = F_\theta(\xi)$, it follows that the angle between $[1,\xi]$ and the unit circle, and the angle between $[1,\omega_{m(n)-1}]$ and the unit circle, cannot both be small. We may assume that there is a uniform $\delta > 0$, such that $\xi \in \Omega^1_{\delta,r}$(In fact, if $\xi \notin \Omega^1_{\delta,r}$ for $\delta > 0$ and $r > 0$ small, from $\arg(\xi/\omega_{m(n)-1}) \approx \pm 2\pi/3$ and $|\xi - 1|/|\omega_{m(n)-1} - 1| \approx 1$, we get $\omega_{m(n)-1} \in \Omega^1_{\delta,r}$, which is what we want to prove). In Figure 5, the angle between the ray R and the unit circle, and the angle between the ray L and the unit circle, are both equal to $\delta\pi$. By Lemma 4.1, Lemma 4.5, Lemma 4.6 and the condition that $r > 0$ is small, it follows that $\xi \in \widehat{U}_{d'}(J_n')$ where $d' > 0$ is some constant dependent only on d. This implies that there exists a uniform $0 < C < \infty$ such that

$$|1-\xi| \le C|J_n'|. \tag{8}$$

In fact, let $H > 0$ be the distance between ξ and the unit circle. Since ξ is above L, it follows that $|1-\xi| < CH$. Since $\xi \in \widehat{U}_{d'}(J_n')$ and d' depends only on d, it follows that $H < C|J_n'|$. The inequality (8) follows.

Now by Lemma 4.6, we may take $e > 0$ such that

$$U_e(\phi(I)) \subset \phi(H_d(I)) \tag{9}$$

for any $I \subset \partial D$. We claim that the angle between the straight segment $[1,\omega_{m(n)-1}]$ and the unit circle cannot be too small. Let us prove this by contradiction. Suppose this were not true. Then there would be an arc $J \subset \partial\Delta$ such that

$$\omega_{m(n)-1} \in U_e(J) \tag{10}$$

and moreover,

$$|J| \ll |1-\omega_{m(n)-1}| \approx |1-\xi|. \tag{11}$$

Let $J = [a,b]$ such that b is the end point which is nearer to 1. Let $J_n' = [c,d]$ such that c is the end point which is nearer to 1. From (11), it follows that

$$|J| \ll |1-b| \approx |1-\omega_{m(n)-1}| \approx |1-\xi| \le C|1-d| \le C'|J_n'|, \tag{12}$$

where $C > 0$ and $C' > 0$ are some uniform constants. Now by Lemma 4.8 and (12), and the quasi-symmetric property of h, we get

$$L(J) \ll L(J_n'). \tag{13}$$

The notation $\ll$ in (13) means that $L(J)/L(J_n')$ can be arbitrarily small provided that the angle between the straight segment $[1, \omega_{m(n)-1}]$ and the unit circle is small enough.

On the other hand, by (9) and (10), we get

$$z_{m(n)-1} \in H_d(\phi^{-1}(J)). \tag{14}$$

From (13), Lemma 4.7 and the f_θ-invariant property of l, we get

$$l(\phi^{-1}(J)) = L(J) \ll L(J_n') = l(J_n) = l(I_{m(n)}). \tag{15}$$

But (14) and (15) contradict with the minimal property of $m(n)$. This proves the lemma in Case 1.

Case 2. $I_{m(n)}$ contains a critical value of f_θ. Let $J = \phi(I_{m(n)})$. It follows that J contains a critical value v of F_θ. We claim that the angle between the straight segment $[v, \omega_{m(n)}]$ and the unit circle has a uniform positive lower bound. Let us first show how the lemma in Case 2 is implied by the claim. In fact, as in the proof of Case 1, one can see that, the two pre-images of $\omega_{m(n)}$ under F_θ, which are close to the unit circle, belong to a small neighborhood of some critical point on the unit circle, say 1. In particular, $\omega_{m(n)-1}$ belongs to this neighborhood. It follows from the claim that the straight segment $[1, \omega_{m(n)-1}]$ and the unit circle has a uniform positive lower bound also. This implies the lemma.

Now let us prove the claim. Suppose it were not true. Then by Lemma 4.9, for any $d' > 0$ and $0 < \eta < 1$, there exists an $m(n)$ and an arc $J' \subset \partial\Delta$ such that

$$\omega_{m(n)} \in U_{d'}(J') \tag{16}$$

and

$$L(J') < \eta L(J). \tag{17}$$

We then transfer this data to the dynamical plane of f_θ by ϕ^{-1}. By Lemma 4.6, we will get a contradiction with the minimal property of $l_{m(n)}$ provided d' and η are small enough. This proves the lemma in Case 2. This finishes the proof of Lemma 4.11. ∎

5. Pull back argument

Now let us prove the Main Theorem. The proof is by contradiction. Assume that K_θ has positive Lebesgue measure. Let $z_0 \in K_\theta$ be a Lebesgue point. Let

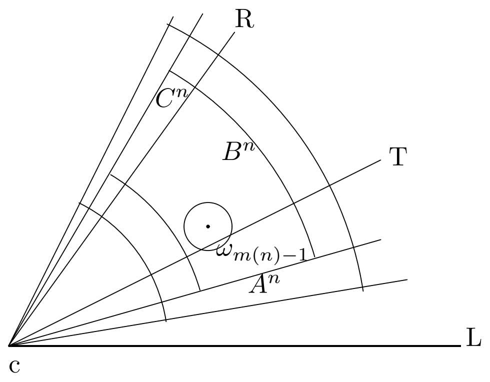

Figure 6. Construction of $(A_n, B_n, C_n, \omega_{m(n)-1})$.

$z_k = f_\theta^k(z_0)$ for $k \geq 0$. By using a standard pull back argument (for example, see Proposition 1.14 [LY]), it follows that z_k approaches to the post-critical set of f_θ. By the main result proved in [Z], the post-critical set of f_θ is the boundary of the Siegel disk which is centered at the origin. Thus we get

Lemma 5.1: *$z_k \to \partial D$ as $k \to \infty$.*

As in §4, let us set $\omega_k = \phi(z_k)$ where $\phi : \mathbb{C} \to \mathbb{C}$ is the quasiconformal homeomorphism defined in Lemma 2.7. It follows that $\omega_k \to \partial\Delta$. Recall that $\Omega_* = \mathbb{C} - \overline{\Delta}$. For a subset $X \subset \Omega_*$, let us use $\mathrm{Diam}_{\Omega_*}(X)$ to denote the diameter of X with respect to the hyperbolic metric $d\rho_*$ of Ω_*. For any subset E of the complex plane, we use $\mathrm{area}(E)$, and $\mathrm{Diam}(E)$ to denote the area and the diameter of E with respect to the Euclidean metric respectively.

Lemma 5.2: *There exist $K_1, K_2, K_3 > 0$ independent of n, such that for every n large enough, there are simply connected domains $C^n \subset B^n \subset A^n \subset \Omega_*$ satisfying*

1. $\omega_{m(n)-1} \in B^n$,
2. $F_\theta(C^n) \subset \Delta$,
3. $\mathrm{mod}\ (A^n - B^n) \geq K_1$,
4. $\mathrm{area}(C^n)/\mathrm{Diam}(B^n)^2 \geq K_2$,
5. $\mathrm{Diam}_{\Omega_*}(A^n) \leq K_3$.

Proof. Let $r > 0$ and $0 < \epsilon < 1/12$ be some small numbers so that Lemma 4.11 holds for all n large enough. By Lemma 2.6, it follows that for $r > 0$ small enough, there are two domains contained in $B_r(c) - \Delta$ which are tangent with the unit circle at c and which are mapped by F_θ into the outside of the unit disk. By Lemma 4.11, we may assume that $\omega_{m(n)-1}$ lies in one of these two domains, say U. Again by Lemma 2.6, for $r > 0$ small enough, there is exactly one domain, say V, which is contained in $B_r(c) - \Delta$ and which is mapped by F_θ into the inside of the unit disk. Let L and R be the two half rays which are tangent with U at c (see Figure 6). When viewed from $\omega_{m(n)-1}$, U is approximately an angle domain formed by the two straight segments which start from c and which lie on R and L, respectively. To simplify notation, we still use R and L to denote them. It follows that the angle between R and L is $\pi/3$. Let T be the straight segment between R and L and which is on the boundary of $\Omega^c_{\epsilon,r}$. By assumption, the angle between T and L is $\epsilon\pi$. For convenience, we use the polar coordinate system formed by (c, L). By Lemma 4.11, $\omega_{m(n)-1} \in \Omega^c_{\epsilon,r}$. Therefore, we have

$$\omega_{m(n)-1} = r_0 e^{\lambda\pi}.$$

for some $\epsilon < \lambda < 1/3$ and $0 < r_0 < r$. Now let A^n be the region bounded by

$$\frac{1}{4}\epsilon\pi \le \theta \le (1/3 + 2\epsilon)\pi,$$

and

$$r_0/2 \le \rho \le 3r_0/2.$$

Let B^n be the region bounded by

$$\frac{1}{2}\epsilon\pi \le \theta \le (1/3 + \epsilon)\pi,$$

and

$$3r_0/4 \le \rho \le 5r_0/4.$$

Let $C^n = B \cap V$. It is not difficult to check that there are uniform constants $K_i > 0, 1 \le i \le 3$ such that for the domains defined above, the properties in the Lemma are all satisfied. We leave the details to the reader. ∎

Let us prove the Main Theorem now. Assume that K_θ is a positive measure set. Set

$$\widetilde{K}_\theta = \phi(K_\theta).$$

It follows that $\widetilde{K}_\theta$ also has positive Lebesgue measure. Since ϕ is quasiconformal, it maps almost every Lebesgue point of K_θ to a Lebesgue point of $\widetilde{K}_\theta$. Thus we may assume that $\omega_0 = \phi(z_0)$ is a Lebesgue point of $\widetilde{K}_\theta$ and z_0 is a Lebesgue point of K_θ.

Now let us consider the pull back of $(A^n, B^n, C^n, \omega_{m(n)-1})$ along the orbit $\{\omega_k\}$. For $1 \le l \le m(n)-1$, let us denote the connected component of $F_\theta^{l-m(n)}(A^n)$ containing ω_{l-1} by A^n_{l-1}. Then A^n_0 is the connected component of $F_\theta^{1-m(n)}(A^n)$ which contains ω_0, and $A^n_{m(k)-1}$ is the connected component of $F_\theta^{m(k)-m(n)}(A^n)$ which contains $\omega_{m(k)-1}$ for $1 \le k < n$. We use B^n_0 and C^n_0 to denote the subdomains of A^n_0 which are the pull backs of B^n and C^n by $F_\theta^{1-m(n)}$. It follows that $C^n_0 \subset B^n_0 \subset A^n_0$.

Since F_θ^{-1} contracts the hyperbolic metric in Ω_*, we have for all $1 \le l \le m(n)-1$,

$$Diam_{\Omega_*}(A^n_{l-1}) \le K_3 \tag{18}$$

where K_3 is the constant in (5) of Lemma 5.2.

By Lemma 4.11, there is an $N > 0$ such that when $k > N$, $\omega_{m(k)-1} \in \Omega^c_{\epsilon,r}$ for some $c \in \{1, -1\}$. Since $\omega_k \to \partial\Delta$ and $m(k) \to \infty$, by (18), there is an N' and an $0 < \eta < 1$ such that for all $k \ge N'$,

$$A^n_{m(k)-1} \subset \Omega^c_{\eta\epsilon,r}. \tag{19}$$

¿From Lemma 3.2 and (19), it follows that there is a $0 < \delta < 1$ independent of n such that for every k with $\max\{N, N'\} \le k \le n$,

$$Diam_{\Omega_*}(A^n_{m(k)-1}) \le (1-\delta) Diam_{\Omega_*}(A^n_{m(k)}). \tag{20}$$

Since $\{m(k)\}$ is an infinite sequence, by (20), it follows that as $n \to \infty$, $Diam(A^n_0) \to 0$ and hence $Diam(B^n_0) \to 0$ as $n \to \infty$. On the other hand, by (3), (4) of Lemma 5.2 and Kobe's distortion theorem, we get a constant $0 < C < \infty$ such that for all n large enough, the following conditions hold:

1. $\omega_0 \in B^n_0$,
2. $C^n_0 \subset B^n_0$ and
3. $areaC^n_0 \ge Cdiam(B^n_0)^2$.

By (2) of Lemma 5.2, $C^n_0 \cap \widetilde{K}_\theta = \emptyset$. This implies that ω_0 is not a Lebesgue point of $\widetilde{K}_\theta$. This is a contradiction. The proof of the Main Theorem is complete.

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